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From asymptotic to finite Heisenberg chain—the evolution of Bethe solutions

W J Caspers[1], M Łabuz[2], A Wal[2], M Kuźma[2] and T Lulek[2]

[1] Bastinglaan 29, 7548 AM Enschede, The Netherlands
[2] Institute of Physics, University of Rzeszow, Rejtana 16a, 35-310 Rzeszow, Poland

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Abstract

Solutions of Bethe equations are found for different numbers of nodes in the Heisenberg chain for $S = 1/2$ and for chosen winding numbers. The computing procedure starts from asymptotic solutions. It is shown that the evolution of solutions has quasi-continuous form even for a wide range of nodes number N. However, it is observed that in some cases critical and limiting points appear.

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1. Introduction

The Bethe ansatz in asymptotic form for the Heisenberg ring in a thermodynamical formulation results in a problem that is easy to solve and widely discussed in the literature [1–6]. Modern technologies (e.g., nanotechnology, nanolithography, spintronics, low-dimensional devices, etc.) require, however, exact solutions for finite cases and even for very small values of N. Usually, to solve the problem for fixed N, admissible sets of winding numbers have to be chosen [1]. In general, for a given chain of length N, the set of solutions cannot immediately be found from that of another length. Nevertheless, it has been observed that for chains of lengths that are close to one another and the same set of winding numbers $\{\lambda_1, \lambda_2, \lambda_3\}$ the solutions are similar. But this regularity is not valid for every length N. For this reason we study here the evolution of solutions for given $\{\lambda_1, \lambda_2, \lambda_3\}$, starting from $N = 1000$, a number for which we take the asymptotic solution to be a good approximation, and ending at a value of N that is as small as possible (even as small as $N = 6$). We have chosen those sets of winding numbers that are present for very small values of N, i.e. the winding numbers for the case $N = 6$, $r = 3$, the one well examined in other papers [7, 9]. For all the sets of winding numbers, we have found solutions of the Bethe equations, for many values of N varying from 6 to 1000.

In section 2 we describe different types of solutions for the Heisenberg chain with N sites and r spin deviations, depending on the choice of winding numbers. Furthermore, the special values of N for which the character of the solutions changes are introduced. These special values are the critical, limiting and transition points.

Section 3 contains solutions for $r = 3$ and for a wide range of values of N (from the asymptotic case up to small integers). In this section the procedure of computations and the form of Bethe equations in dependence on sets $\{\lambda_l\}$ have been introduced. Our method relies on fixing the sequence of winding numbers, and thus also the total number r of reversed spins is fixed. Clearly, it yields the change of quasimomentum of appropriate Bethe eigenstates for different values of N. Thus our treatment of the limit $N \to \infty$ implies $k \to 0$, i.e. quasimomentum tends towards the centre of the Brillouin zone, which differs from the standard regime (r/N fixed for $N \to \infty$). But our approach allows us to obtain effectively proper excited states for a given finite N.

Section 4 contains the results for $N = 6$, $r = 3$ and a discussion of these results and general conclusions are collected in the final section.

2. Bethe equations and types of solutions

2.1. Physics of the Heisenberg chain

The Heisenberg chain contains identical particles with spin $\frac{1}{2}$, with nearest-neighbour interaction. The Hamiltonian [1, 2] of this system has the form

$$\hat{H} = \sum_{n=1}^{N} (4\mathbf{S}^n \mathbf{S}^{n+1} - 1) \tag{1}$$

where $\mathbf{S}^n$ refers to the spin vector at position n along the chain. Furthermore, the identification

$$\mathbf{S}^{N+1} \equiv \mathbf{S}^1 \tag{2}$$

ensures periodic boundary conditions.

A stationary state of this system is given by the Bethe solution [1]:

$$\sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} \sum_{P} \mathrm{e}^{\mathrm{i}(\sum_{l=1}^{r} k_{P(l)} n_l + \frac{1}{2} \sum_{j<l} \varphi_{P(j)P(l)})} |n_1 n_2 \ldots n_r\rangle \tag{3}$$

where P is a permutation of the positions of the r inversions or pseudoparticles, and the sum runs over all such permutations to assure that the pseudoparticles are indistinguishable. The symbol $|n_1 n_2 \ldots n_r\rangle$ denotes the magnetic configuration corresponding to r inversions on the sites $n_1, n_2, \ldots, n_r$ of the lattice.

In the general case with r deviations, one can consider them as pseudoparticles that move freely along the chain, with the exception of a scattering or collision. A scattering only takes place if two pseudoparticles occupy neighbouring positions on the chain and thus it is a two-body short-range interaction. The two pseudoparticles l and j move with their pseudomomenta k_l and k_j until they reach neighbouring positions. Here scattering or exchange of momenta takes place, and is accompanied by a phase change $\varphi_{l,j}$. Pseudomomenta k_l and k_j are related to phases $\varphi_{l,j}$ by the *reflection condition*:

$$2\cot\frac{\varphi_{l,j}}{2} = \cot\frac{k_l}{2} - \cot\frac{k_j}{2} \qquad \varphi_{l,j} = -\varphi_{j,l}. \tag{4}$$

Pseudomomenta and related phases should also satisfy the boundary conditions

$$N k_l = 2\pi\lambda_l + \sum_{j \neq i} \varphi_{l,j} \qquad l = 1, 2, \ldots, r \tag{5}$$

where N is the length of the chain and λ_l are winding numbers, which satisfy the condition

$$-\frac{N}{2} \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \frac{N}{2}. \tag{6}$$

Equations (4) and (5) are Bethe–Hulthén (BH) equations.

The energy of the eigenstate (3) is given by

$$E = -4\sum_{l=1}^{r}(1 - \cos k_l). \tag{7}$$

2.2. Types of solutions

The key role in the classification of the Bethe–Hulthén solutions is played by winding numbers $\{\lambda_l\}$. According to the relations between these numbers, one may distinguish between different types of solutions. It implies different forms of BH equations. Clearly, the general form of the system of Bethe equations is unique. The form of solutions, however, varies for different sequences of winding numbers. Anticipating such solutions, we adapt the particular form of the system of Bethe equations. In the special case ($N = 6$, $r = 3$) which we discuss in detail in this paper, there are three winding numbers, which are related by the following formula:

$$-3 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant 3. \tag{8}$$

There are four main types of sets of winding numbers that give different kinds of solutions [9]:

- (I) Consecutive λ_l differ at least by 2. In this case all parameters k_l and $\varphi_{l,j}$ are real.
- (IIa) Two consecutive winding numbers are equal. For this set of $\{\lambda_l\}$ only one pseudomomentum is real, while the other two are complex conjugated. For this case there exists a *limiting point* (N_{lim}) as will be explained in the next section.
- (IIb) Two consecutive winding numbers differ by 1. For such a set of $\{\lambda_l\}$ critical (N_{cr}) and limiting points (N_{lim}) exist [7, 11] (see next section).
- (III) All winding numbers are equal. We assume that one pseudomomentum is *real* and the other two are *complex conjugated*.

2.3. Critical and limiting points

Critical and limiting points (N_{cr} and N_{lim}, respectively) appear in type (IIb) solutions. In that case, starting from asymptotic solutions the parameters are real, up to the point where N reaches its critical value N_{cr}. At this point a pair of real wavenumbers (k_l, k_j) changes into a complex conjugated pair and the corresponding phase $\varphi_{l,j}$ will also assume a complex value.

Such a complex solution exists below N_{cr} until N reaches another characteristic value: N_{lim}. Then the imaginary part of the two wavenumbers will diverge and the corresponding N is the lowest value for which this solution should be taken into consideration. Below this limit it coincides with a solution for another parameter set.

For solutions of type (IIa) only limiting points occur, because the parameters are complex in the full range of N values. We do not observe a critical point in this case.

3. Solutions for $r = 3$

3.1. Procedure of computations

The procedure for the solution of the Bethe equations (4) and (5), for given winding number, starts from an asymptotic solution for large N, typically $N = 1000$. As a rule the character of the solution does not change drastically in lowering N and the values of the parameters show a quasi-continuous behaviour, apart from the case that N reaches one of those special points discussed in subsection 2.3. The critical point as a rule corresponds to a non-integer N, which separates two regions of quantitatively different behaviour of the solution. Further we introduce each solution based on relations between succeeding winding numbers $\{\lambda_l\}$.

3.2. The set of winding numbers: $\lambda_1 \leqslant \lambda_2 - 2$, $\lambda_2 \leqslant \lambda_3 - 2$

In this case the consecutive winding numbers differ by at least 2. All solutions obtained are real, i.e. all pseudomomenta and phases are real numbers. In accordance with the procedure described in subsection 3.1, the first step in the calculations is the determination of the results for $N = 1000$, considered to be described by the asymptotic case in a fairly good approximation. For the set of sufficiently small winding numbers, the solutions are characterized by parameters

$$k_l \sim \frac{2\pi\lambda_l}{N} \qquad \varphi_{l,j} \sim -\frac{4\pi\lambda_j}{(\lambda_l - \lambda_j)N} \qquad l \neq j \quad l, j = 1, 2, 3. \tag{9}$$

Solutions for smaller N are obtained from Bethe–Hulthén equations:

$$Nk_1 = 2\pi\lambda_1 + \varphi_{1,2} + \varphi_{1,3} \tag{10a}$$

$$Nk_2 = 2\pi\lambda_2 - \varphi_{1,2} + \varphi_{2,3} \tag{10b}$$

$$Nk_3 = 2\pi\lambda_3 - \varphi_{1,3} - \varphi_{2,3} \tag{10c}$$

$$\sin(\varphi_{1,2})[\cos(k_1 - k_2) - 1] + [\cos(\varphi_{1,2}) + 1][\sin(k_1) - \sin(k_2) - \sin(k_1 - k_2)] = 0 \tag{10d}$$

$$\sin(\varphi_{1,3})[\cos(k_1 - k_3) - 1] + [\cos(\varphi_{1,3}) + 1][\sin(k_1) - \sin(k_3) - \sin(k_1 - k_3)] = 0 \tag{10e}$$

$$\sin(\varphi_{2,3})[\cos(k_2 - k_3) - 1] + [\cos(\varphi_{2,3}) + 1][\sin(k_2) - \sin(k_3) - \sin(k_2 - k_3)] = 0. \tag{10f}$$

Results for successive $N < 1000$ show the expected quasicontinuous behaviour. Here we only give the results for $N = 6$, $r = 3$, which are presented in table 1. In this table, the symbol I denotes the case in which all three pseudomomenta are non-vanishing, whereas $\mathrm{I}^{(1)}$, $\mathrm{I}^{(2)}$ and $\mathrm{I}^{(3)}$, respectively, correspond to the cases in which one, two and three pseudomomenta are equal to zero.

A typical example is shown in figure 1, i.e. the case $\{\lambda_1, \lambda_2, \lambda_3\} = \{-3, -1, 1\}$. In this figure the evolution of the solutions within the range $6 \leqslant N \leqslant 1000$ is shown. In the asymptotic regime the changes of parameters $\{k_1, k_2, k_3, \varphi_{1,2}, \varphi_{1,3}, \varphi_{2,3}\}$ as functions of N are quasicontinuous. In a few cases some parameters exist only for $N > N_t$ (N_t being a transition point), e.g. $N_t = 8$ for $\{-3, 1, 3\}$ (see figure 2). Below this point, one of the parameters (pseudomomentum or phase) leaves the $(-\pi, \pi)$ range.

3.3. The set of winding numbers: $\lambda_1 \leqslant \lambda_2 - 2$, $\lambda_2 = \lambda_3 - 1$

In this case there exists a pair λ_n, λ_{n+1} of consecutive nonzero λ, which differ by 1, i.e. $\lambda_{n+1} = \lambda_n + 1$. It implies a change of the form of Bethe–Hulthén equations. Let us introduce the notation

$$\begin{aligned} &k_n = A_n - b_n \qquad && k_{n+1} = A_n + b_n \qquad && \varphi_{l,n} = P_{l,n} - q_{l,n} \\ &\varphi_{l,n+1} = P_{l,n} + q_{l,n} \qquad && \varphi_{n,n+1} = \pi - q_n && \end{aligned} \tag{11}$$

and

$$P_{l,n} = -P_{n,l} \qquad \text{and} \qquad q_{l,n} = -q_{n,l}. \tag{12}$$

For the case $N = 6$ and $r = 3$ we have $n = 1$ or $n = 2$ and consequently $l = 3$ or $l = 1$, respectively.

Table 1. Results for $N = 6$, $r = 3$.

Type of solution	No	$\{\lambda_l\}$	k_1	k_2	k_3	$\varphi_{1,2}$	$\varphi_{1,3}$	$\varphi_{2,3}$	k	E
I	1	(−3, −1, 1)	π	−1.72	1.72	2.33	−2.33	−1.72	3	−2.80
		(−1, 1, 3)	−1.72	1.72	π	−1.72	−2.33	2.33	3	−2.80
$\mathrm{I}^{(1)}$	2	(−3, −1, 0)	−2.77	−1.42	0	2.23	0	0	2	−1.28
	3	(−2, 0, 1)	−2.39	0	1.34	0	−1.76	0	−1	−1.00
	4	(−2, 0, 2)	−2.51	0	2.51	0	−2.51	0	0	−2.19
	5	(−1, 0, 0)	−1.05	0	0	0	0	0	−1	1.00
	6	(−1, 0, 1)	−1.26	0	1.26	0	−1.26	0	0	0.19
	7	(−1, 0, 2)	−1.34	0	2.39	0	−1.76	0	1	−1.00
	8	(0, 0, 1)	0	0	1.05	0	0	0	1	1.00
	9	(0, 1, 3)	0	1.42	2.77	0	0	2.23	−2	−1.28
$\mathrm{I}^{(2)}$	10	(−3, 0, 0)	$-\pi$	0	0	0	0	0	3	−0.50
		(0, 0, 3)	0	0	π	0	0	0	3	−0.50
	11	(−2, 0, 0)	−2.09	0	0	0	0	0	−2	0
	12	(0, 0, 2)	0	0	2.09	0	0	0	2	0
$\mathrm{I}^{(3)}$	13	(0, 0, 0)	0	0	0	0	0	0	0	1.50
II(a)	14	(−2, 1, 1)	$-\pi$	$1.57 - \mathrm{i}\infty$	$1.57 + \mathrm{i}\infty$	$-\pi + 1.10\mathrm{i}$	$-\pi - 1.10\mathrm{i}$	$-\mathrm{i}\infty$	0	$-1.50^{(1)}$
		(−1, −1, 2)	$-1.57 - \mathrm{i}\infty$	$-1.57 + \mathrm{i}\infty$	π	$-\mathrm{i}\infty$	$-\pi - 1.10\mathrm{i}$	$-\pi + 1.10\mathrm{i}$	0	$-1.50^{(1)}$
	15	(−1, −1, 0)	−1.05 − 0.73i	−1.05 + 0.73i	0	−4.40i	0	0	−2	0.78
	16	(−1, −1, 1)	−1.51 − 2.80i	−1.51 + 2.80i	1.90	−14.76i	−2.17 − 0.87i	−2.17 + 0.87i	−1	−050
	17	(−1, 1, 1)	−1.90	1.51 − 2.80i	1.51 + 2.80i	−2.17 + 0.87i	−2.17 − 0.87i	−14.76i	1	−0.50
	18	(0, 1, 1)	0	1.05 − 0.73i	1.05 + 0.73i	0	0	−4.40i	2	0.78
II(b)	19	(−2, −1, 0)	$-1.57 + \mathrm{i}\infty$	$-1.57 - \mathrm{i}\infty$	0	$\pi + \mathrm{i}\infty$	0	0	3	$0.50^{(2)}$
		(0, 1, 2)	0	$1.57 + \mathrm{i}\infty$	$1.57 - \mathrm{i}\infty$	0	0	$\pi + \mathrm{i}\infty$	3	$0.50^{(2)}$
III	20	(−1, −1, −1)	$-\pi$	−1.09i	1.09i	5.44i	−5.44i	−1.09i	3	0.80*
		(1, 1, 1)	−1.09i	1.09i	π	−1.09i	−5.44i	5.44i	3	0.80*

Quasimomentum k is given in units $2\pi/N$, E is given in units J and rescaled. The solutions marked by $^{(1)}$ and $^{(2)}$ contain values of b_n and q_n obtained using limit $N \to N_{\mathrm{lim}}$ considerations. $^{(*)}$ denotes results obtained for changed set of winding numbers (see subsection 3.5).

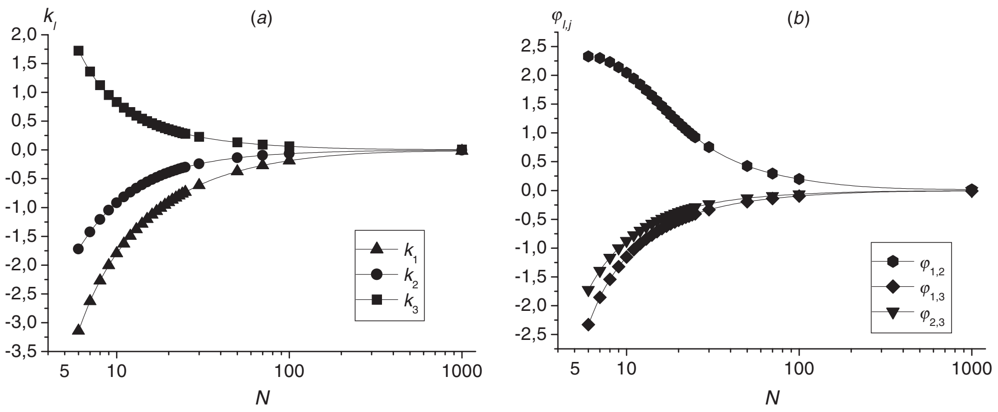

Figure 1. Evolution of solutions for set (−3, −1, 1) as the representative of the I type: (*a*) k_l, (*b*) $\varphi_{l,j}$.

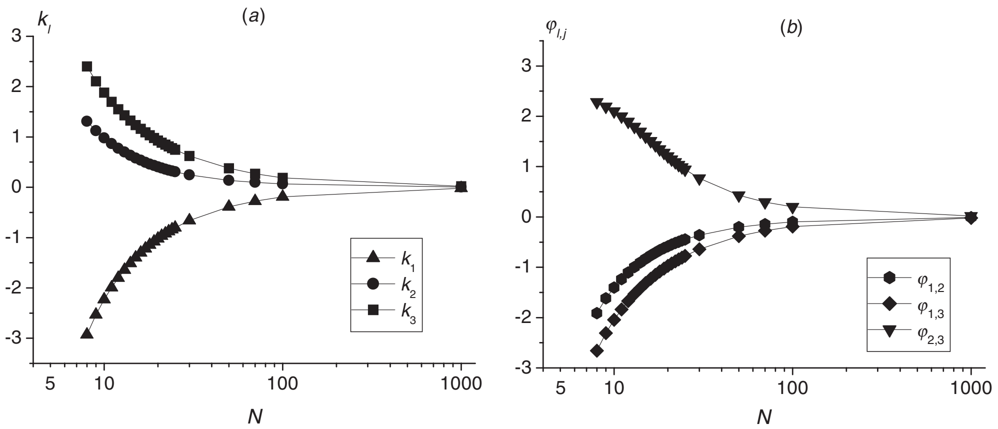

Figure 2. Evolution of solutions for set (−3, 1, 3): (*a*) evolution of k_l, (*b*) evolution of $\varphi_{l,j}$. Transition point $N_t = 8$.

Then Bethe–Hulthén equations (4) and (5) take on the form

$$Nk_l = 2\pi\lambda_l + \sum_{j\neq l} \varphi_{l,j} \tag{13a}$$

$$NA_n = \pi(2\lambda_n + 1) + \sum_{l(\neq n,n+1)} P_{n,l} \tag{13b}$$

$$Nb_n = q_n + \sum_{l(\neq n,n+1)} q_{n,l} \tag{13c}$$

$$\cos\left(\frac{q_n}{2}\right)\sin(b_n) + \sin\left(\frac{q_n}{2}\right)[\cos(A_n) - \cos(b_n)] = 0 \tag{13d}$$

$$\begin{aligned}\sin(-P_{l,n})[\cos(A_n - k_l) - \cos(b_n)] + [\cos(-P_{l,n}) + \cos(-q_{l,n})]\\ \times [\sin(A_n) - \sin(k_l)\cos(b_n) - \sin(A_n - k_l)] = 0\end{aligned} \tag{13e}$$

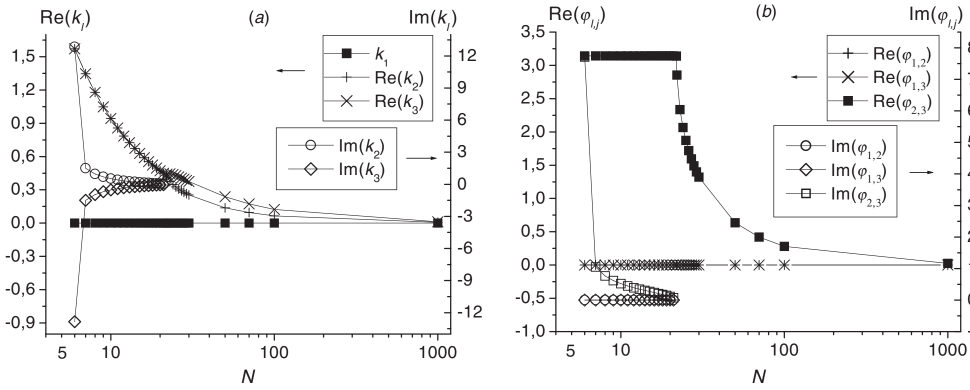

Figure 3. Evolution of solutions for set (0, 1, 2) as the representative of the II(b) type: (*a*) k_l, (*b*) $\varphi_{l,j}$. Critical point $N_{\mathrm{cr}} = 21.86$.

$$\sin(-q_{l,n})[\cos(A_n - k_l) - \cos(b_n)] + [\cos(-P_{l,n}) + \cos(-q_{l,n})] \times [-\cos(k_l)\sin(b_n) + \sin(b_n)] = 0 \tag{13f}$$

where $l \neq n, n+1$.

We observe the existence of a critical point N_{cr}; the solutions are real for $N > N_{\mathrm{cr}}$ and complex for $N < N_{\mathrm{cr}}$. For the complex regime we put

$$k_n = A_n - \mathrm{i}b_n \qquad k_{n+1} = A_n + \mathrm{i}b_n \qquad \varphi_{l,n} = P_{l,n} - \mathrm{i}q_{l,n}$$
$$\varphi_{l,n+1} = P_{l,n} + \mathrm{i}q_{l,n} \qquad \varphi_{n,n+1} = \pi - \mathrm{i}q_n. \tag{14}$$

Then equations (13*d*)–(13*f*) take on the form

$$\cosh\left(\frac{q_n}{2}\right)\sinh(b_n) + \sinh\left(\frac{q_n}{2}\right)[\cos(A_n) - \cosh(b_n)] = 0 \tag{13g}$$

$$\sin(-P_{l,n})[\cos(A_n - k_l) - \cosh(b_n)] + [\cos(-P_{l,n}) + \cosh(-q_{l,n})] \times [\sin(A_n) - \sin(k_l)\cosh(b_n) - \sin(A_n - k_l)] = 0 \tag{13h}$$

$$\sinh(-q_{l,n})[\cos(A_n - k_l) - \cosh(b_n)] + [\cos(-P_{l,n}) + \cosh(-q_{l,n})] \times [-\cos(k_l)\sinh(b_n) + \sinh(b_n)] = 0. \tag{13i}$$

Another characteristic feature for that case is the appearance of a limiting point N_{lim}, i.e. the smallest value of N for which a solution can be found. In a close neighbourhood of N_{lim} we can observe a drastic change of value for b_n and q_n and for $N \to N_{\mathrm{lim}}$ there exists the limit $b_n \to \infty$ and $q_n \to \infty$. Furthermore, from (13*g*) we can observe that $A_n \to \pm\pi/2$ in the limiting case. For $N = 6$, $r = 3$ only one set of $\{\lambda_l\}$ of the type described in this subsection gives a solution (see table 1 and figure 3). The values of b_n and q_n written in table 1 come from theoretical considerations of the limit $N \to N_{\mathrm{lim}}$, because the numerical procedure fails for this point.

3.4. The set of winding numbers: $\lambda_1 < \lambda_2$, $\lambda_2 = \lambda_3$

In this set of $\{\lambda_l\}$, two neighbouring winding numbers are equal. Following the solutions starting from large N, it can be observed that the two wavenumbers connected with the

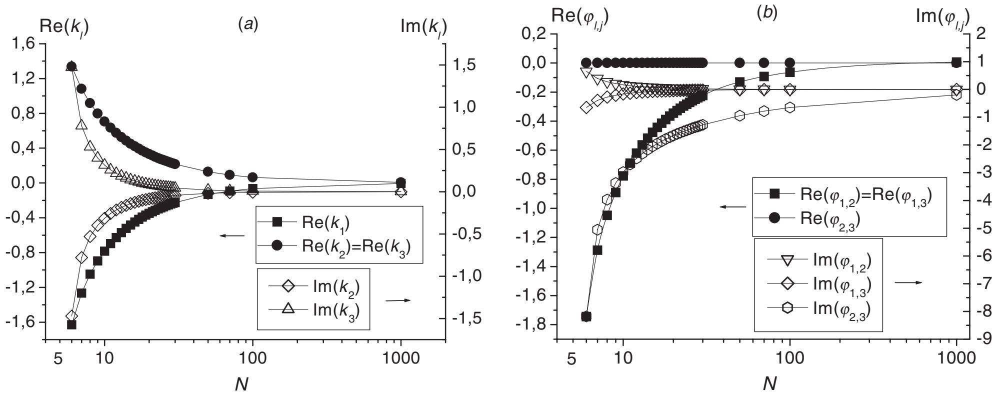

Figure 4. Evolution of solutions for set $(-1, 1, 1)$ as the representative of the II(a) type: (a) k_l, (b) $\varphi_{l,j}$.

degenerate winding numbers ($\lambda_n = \lambda_{n+1}$) are complex conjugated. Denoting the third one by the index l we make the following substitutions:

$$\begin{aligned} &k_n = A_n - \mathrm{i}b_n \qquad k_{n+1} = A_n + \mathrm{i}b_n \qquad \varphi_{l,n} = P_{l,n} - \mathrm{i}q_{l,n} \\ &\varphi_{l,n+1} = P_{l,n} + \mathrm{i}q_{l,n} \qquad \varphi_{n,n+1} = -\mathrm{i}q_n. \end{aligned} \tag{15}$$

Formulae (14) are in fact the same as (15) except for the last equation, where there is a difference (of π) in $\varphi_{n,n+1}$, which denotes the phase between two particles with degenerate winding numbers. Thus some equations for the pseudomomenta and phases should be modified: (13b) and (13g) should respectively be replaced by

$$NA_n = 2\pi\lambda_n + \sum_{l(\neq n,n+1)} P_{n,l} \tag{16a}$$

and

$$\sinh\left(\frac{q_n}{2}\right)\sinh(b_n) + \cosh\left(\frac{q_n}{2}\right)[\cos(A_n) - \cosh(b_n)] = 0. \tag{16b}$$

In this case there are no critical points, therefore for every N, the solution takes the form (15) with a real k_l.

All solutions for this set of $\{\lambda_l\}$ for $N = 6$ are contained in table 1. The dependence on N of this type of solution is also shown in figure 4, for a typical example.

Starting from the asymptotic solutions for large N one may arrive at the solutions for smaller values of N by the process of continuations, given in subsection 3.2. Now the formulae are more complicated than those used in previous examples. For nondegenerate λ_l we have

$$k_l \sim \frac{2\pi\lambda_l}{N} \tag{17a}$$

and for degenerate λ_n

$$A_n \sim \frac{2\pi\lambda_n}{N} \qquad b_n \sim \frac{2\pi\lambda_n}{N^{3/2}} \qquad q_n \sim \frac{2\pi\lambda_n}{N^{1/2}} \tag{17b}$$

while for mixed cases

$$P_{n,l} \sim -\frac{4\pi\lambda_n\lambda_l}{N} \qquad q_{n,l} \sim \frac{4\pi\lambda_n\lambda_l^2}{(\lambda_n - \lambda_l)^2 N^{3/2}} \tag{17c}$$

where $P_{n,l} = -P_{l,n}$ and $q_{n,l} = -q_{l,n}$. The mode l is supposed to be nondegenerate.

3.5. Set of winding numbers: $\lambda_1 = \lambda_2 = \lambda_3$

This case is the most complicated one. First we assume that one quasimomentum is *real* and the others form a *complex conjugated* pair. We use the same indices as in the example of subsection 3.4, i.e., k_l is a real parameter. The other parameters now have the form

$$\begin{aligned} &k_n = A_n - \mathrm{i}b_n \qquad k_{n+1} = A_n + \mathrm{i}b_n \qquad \varphi_{l,n} = P_{l,n} - \mathrm{i}p_{l,n} \\ &\varphi_{l,n+1} = P_{l,n} + \mathrm{i}p_{l,n} \qquad \varphi_{n,n+1} = -\mathrm{i}r_n. \end{aligned} \tag{18}$$

Comparing formulae (15) and (18) one observes a difference in notation ($p_{l,n}$ and r_n instead of $q_{l,n}$ and q_n, respectively). These changes are introduced because of the form of asymptotic expressions for the given set of winding numbers in this case:

$$\begin{aligned} &k_l \sim \frac{2\pi\lambda}{N} + \frac{\kappa}{N^2} \qquad A_n \sim \frac{2\pi\lambda}{N} + \frac{\alpha}{N^2} \qquad b_n \sim \frac{b_0}{N^2} \\ &P_{l,n} \sim \frac{P_0}{N} \qquad p_n \sim \frac{p_0}{N^{1/2}} \qquad r_{l,n} \sim \frac{r_0}{N^{1/2}}. \end{aligned} \tag{19}$$

To find quantities κ, α, b_0, P_0, p_0, r_0 it is necessary to solve the following equations:

$$b_0 = -p_0 + r_0 \tag{20a}$$

$$\kappa + 2\alpha = 0 \tag{20b}$$

$$-\tfrac{1}{2}P_0 b_0^2 + 12\pi^2\lambda^2\alpha + 2\pi\lambda b_0^2 = 0 \tag{20c}$$

$$p_0 b_0^2 = -8\pi^2\lambda^2 \tag{20d}$$

$$r_0 b_0 = 4\pi^2\lambda^2 \tag{20e}$$

$$\alpha = -P_0. \tag{20f}$$

After solving these equations it turns out that b_0 has two equivalent solutions that differ in sign: $\pm b_0$, involving a doubling of the solutions for p_0 and r_0 ($\pm p_0$ and $\pm r_0$). To remove this ambiguity, we choose $+b_0$ and use it to get unique values of p_0 and r_0.

To find the solutions starting from large N, the following equations have to be solved:

$$Nk_l = 2\pi\lambda + 2P_{l,n} \tag{21a}$$

$$NA_n = 2\pi\lambda - P_{l,n} \tag{21b}$$

$$Nb_n = r_n - p_{l,n} \tag{21c}$$

$$\sinh\left(\frac{r_n}{2}\right)\sinh(b) + \cosh\left(\frac{r_n}{2}\right)[\cos(A_n) - \cosh(b_n)] = 0 \tag{21d}$$

$$\begin{aligned} &\sin(-P_{l,n})[\cos(A_n - k_l) - \cosh(b_n)] + [\cos(-P_{l,n}) + \cosh(-p_{l,n})] \\ &\qquad \times [\sin(A_n) - \sin(k_l)\cosh(b_n) - \sin(A_n - k_l)] = 0 \end{aligned} \tag{21e}$$

$$\begin{aligned} &\sinh(-p_{l,n})[\cos(A_n - k_l) - \cosh(b_n) + [\cos(-P_{l,n}) + \cosh(-p_{l,n})] \\ &\qquad \times [-\cos(k_l)\sinh(b_n) + \sinh(b_n)] = 0. \end{aligned} \tag{21f}$$

Considering the case $N = 6$, $r = 3$ we notice that such a solution exists only for the set of winding numbers $(-1, -1, -1)$ (equivalent to $(1, 1, 1)$). Inspecting the changes in parameters describing solutions, starting from asymptotic case, we find that equations (21) can be solved only for $N \geqslant 8$. This value of N turned out to be the last number in the series of N for which

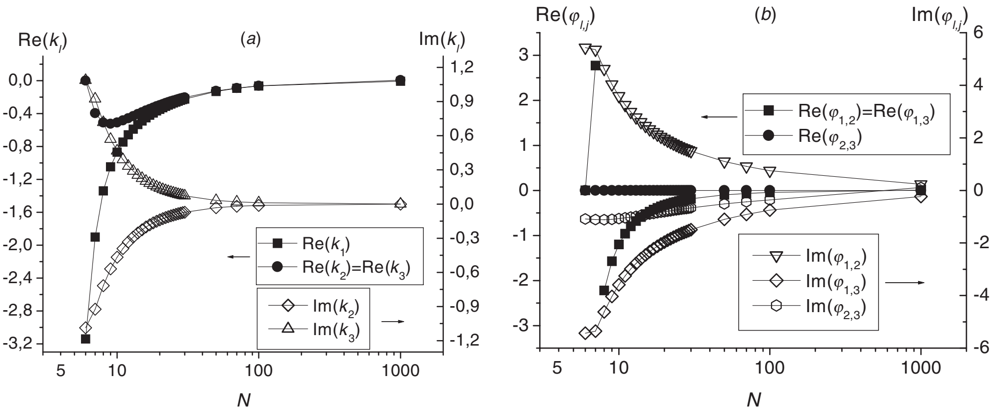

Figure 5. Evolution of solutions for set $(-1, -1, -1)$ as the representative of the III type: (a) k_l, (b) $\varphi_{l,j}$. Solutions for $N = 7$ and $N = 6$ are obtained for a new type of $\{\lambda\}$ (see subsection 3.5).

solutions could be obtained in the interval $(-\pi, \pi)$. For smaller N one of the parameters exceeds this range, but we found a way to construct solutions that obey this condition. It can be shown that a change of the set of winding numbers is possible. Having a set $\{\lambda_1, \lambda_2, \lambda_3)$ we can obtain another one by the following transformation [11]:

$$\lambda_1' = \lambda_1 \mp 2 \qquad \lambda_2' = \lambda_2 \pm 1 \qquad \lambda_3' = \lambda_3 \pm 1. \tag{22}$$

Due to this method, a parameter that was outside the prescribed range is now brought back from the opposite side of the range (see figure 5(b)).

In the case being discussed, we make the transformation $(-1, -1, -1) \Rightarrow (-3, 0, 0)$ (equivalently $(1, 1, 1) \Rightarrow (0, 0, 3)$). Now equations (21) cannot be applied to this case, as the case $(-3, 0, 0)$ (equivalent to $(0, 0, 3)$) belongs to the $\mathrm{I}^{(2)}$ type. Therefore to find solutions for $N = 6$ or $N = 7$ we have to use equations (10) and remember that we look for parameters different from zero, as that took place for results of $\mathrm{I}^{(2)}$ type.

We can immediately observe that the set $\{\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0\}$ results in the solutions $\{k_1 = 0, k_2 = 0, k_3 = 0, \varphi_{1,2} = 0, \varphi_{1,3} = 0, \varphi_{2,3} = 0\}$, so all parameters satisfy all relevant conditions.

4. Results and discussion

In this paper the evolution of solutions for a Heisenberg ring with $r = 3$ spin deviations is analysed. The starting point of the computations was the asymptotic case ($N \to \infty$). Then the number of sites was decreased to a very small number: $N = 6$.

Table 1 contains exact results for the case $N = 6$, $r = 3$. But in cases $^{(1)}$ and $^{(2)}$ using numerical computations we were not able to find solutions for $N = 6$ and the limit values for b_n and q_n were put instead. Furthermore '$*$' means that results were obtained for another set of winding numbers (see subsection 3.5).

Looking at table 1 one can observe equivalent cases, i.e., for the symmetric set of winding numbers ($\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{-\lambda_3, -\lambda_2, -\lambda_1\}$), related sets of the parameters $\{k_l\}$ and $\{\varphi_{l,j}\}$ are obtained, the difference being only in the sign of an individual parameter. The total pseudomomentum—quasimomentum plays the key role here. In table 1 the results are given in units $2\pi/N$. If these are different for symmetric cases then the solutions have to

Table 2. Equivalent cases for $N = 6$, $r = 3$.

Type of solution	Equivalent cases from table 1
I	—
$I^{(1)}$	2 and 9 3 and 7 5 and 8
$I^{(2)}$	11 and 12
$I^{(3)}$	—
II(a)	15 and 18 16 and 17
II(b)	—
III	—

be considered to be essentially different ones. Otherwise solutions obtained for equivalent sets of winding numbers with equal quasimomentum should be considered as identical.

Equivalent solutions for $N = 6$, $r = 3$ are contained in table 2.

5. Conclusions

In this paper we have proposed a method for the calculation of BH solutions. It is based on the determination of solutions for a gradual decrease of N starting from the quasiasymptotic case (N large, e.g., $N = 1000$). For this case, quantum numbers describing the solutions (pseudomomenta and phases) are small (almost equal to zero). This follows from the asymptotic formulae for these numbers, in which N appears in the denominator. With the decrease of the number of sites N the values of these quantum numbers become gradually larger and even in some cases of small N exceed the first Brilluoin zone. For the asymptotic case we have a large number of solutions, determined by the different sets of $\{\lambda_l\}$. Therefore we restricted our considerations to those sets which are expected to give solutions for $N = 6$, the final number of sites we are interested in. The number of solutions also depends on the number of spin deviations r. We have chosen $r = 3$, as a nontrivial case also for $N = 6$. All these solutions are collected in table 1.

Solving the BH equations by direct numerical methods results in a great number of solutions from which some should be removed as a consequence of equivalency or non-physical nature. Our method is promising, because we eliminate additional solutions through accurate determination of the range in which solutions should be obtained.

It is very surprising that for many cases the wavenumbers and phases of the solutions are quasi-continuous functions of the chain length N even in the range for small N (e.g. $(-3, -1, 3)$, figure 1). However, there are cases for which continuity is broken by critical, limiting and transition points (e.g. figure 3). In the case $N = 6$, $r = 3$ there are several equivalent solutions reflecting BH equation symmetry, which can be observed after close inspection of formulae (4) and (5).

Our investigations clearly demonstrate that evolution of solutions is quasi-continuous, with the exception of special cases, such as critical, limiting and transition points. These kinds of nonanalyticity are associated with our choice of tracking the solutions by specification of winding numbers.

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